Projective modules over discrete Hodge algebras

Manoj Kumar Keshari

Department of Mathematics, IIT Mumbai, Mumbai - 400076, India; keshari@math.iitb.ac.in

1 Introduction

All the rings are assumed to be commutative Noetherian and all the modules are finitely generated.

Let A be a ring. In ([6], Theorem 1.1), Vorst proved that if all projective modules over polynomial extensions of A are extended from A, then all projective modules over discrete Hodge A-algebras are extended from A (An A-algebra R is a *discrete Hodge A-algebra* if $R = A[X_0, \ldots, X_n]/I$, where I is an ideal generated by monomials). In this note, we extend the above result of Vorst by proving the following result.

Theorem 1.1 *Let A be a ring and $r > 0$ be an integer. Assume that all projective modules of rank r over polynomial extensions of A are extended from A. Then all projective modules of rank r over discrete Hodge A-algebras are extended from A.*

We note that Lindel gave another proof of Vorst's result ([1], Theorem 1.5) and a proof of (1.1) is implicit in Lindel's proof. But the idea of our proof is different from Lindel's and it also gives other results which we describe below.

Let A be a ring of dimension d and let $r > d/2$. Assume that A is of finite characteristic prime to $r!$. In ([5], Theorem 5), Roitman proved that if P is a projective module of rank r over $R = A[X_1, \ldots, X_n]$ such that $P \oplus R$ is extended from A, then P is extended from A. In particular, if A is a local ring of dimension d, characteristic of A is positive and prime to $d!$, then all stably free modules of rank $> d/2$ over polynomial extensions of A are free.

We will prove the following analogue of Roitman's result for discrete Hodge A-algebras.

Theorem 1.2 *Let A be a ring of dimension d. Assume A is of finite characteristic prime to $r!$. Let R be a discrete Hodge A-algebra and let P be a projective R-module of rank $r > d/2$. If $P \oplus R$ is extended from A, then P is extended from A.*

As a corollary to the above result, if A is a local ring of dimension d, characteristic of A is finite and prime to $d!$, then all stably free modules of rank $> d/2$ over discrete Hodge A-algebras are free.

Now, we will describe our last result. Let A be a ring of dimension d and let $R = A[X_1, \ldots, X_n]$. In ([7], Section 4), Wiemers asked the following question: Is the natural map $\text{Um}_r(R) \to \text{Um}_r(R/(X_1 X_2 \ldots X_k))$ surjective for all r and $1 \le k \le n$?

Wiemers ([7], Proposition 4.1) answered the above question in affirmative when $r \ge d + 2$ or $r = d + 1$ and $1/d! \in A$. We will prove the following result which gives a partial answer to Wiemers question in affirmative.

Theorem 1.3 *Let A be a ring of dimension d. Assume characteristic of A is positive and prime to $d!$. Let $R = A[X_1, \ldots, X_n]$ and let $I \subset J$ be two ideals of R generated by square free monomials. Then the map $\mathrm{Um}_r(R/I) \to \mathrm{Um}_r(R/J)$ is surjective for $r \geq \frac{d}{2} + 2$.*

2 Preliminaries

Given a cartesian diagram of rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_1 \arrow[d, "j_1"'] \\ A_2 \arrow[r, "j_2"'] & A_0 \end{tikzcd}$$

where j_2 is a surjective map. If P is a projective A-module, then the above diagram induces a cartesian diagram ([2], Section 2)

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P_1 \arrow[d] \\ P_2 \arrow[r] & P_0 \end{tikzcd}$$

where $P_i = P \otimes A_i$ for $i = 0, 1, 2$.

We begin by stating the following two results of A. Wiemers ([7], Proposition 2.1 and Theorem 2.3) respectively.

Proposition 2.1 *Given a cartesian square of rings with j_2 surjective and a projective A-module P. Then*

(i) If $\mathrm{Aut}_{A_2}(P_2) \to \mathrm{Aut}_{A_0}(P_0)$ is surjective, then so is $\mathrm{Aut}_A(P) \to \mathrm{Aut}_{A_1}(P_1)$.

(ii) If $\mathrm{Aut}_{A_2}(P_2) \to \mathrm{Aut}_{A_0}(P_0)$ is surjective and $Q \otimes_A A_i \xrightarrow{\sim} P_i$, $i = 1, 2$ for another projective A-module Q, then $P \xrightarrow{\sim} Q$. In particular, if P_1 and P_2 have the cancellation property, then so does P.

(iii) Let, in addition, j_1 be surjective. If $\mathrm{Um}(P_2) \to \mathrm{Um}(P_0)$ is surjective, then so is $\mathrm{Um}(P) \to \mathrm{Um}(P_1)$.

Theorem 2.2 *Let A be a ring and let J be an ideal of $R = A[X_1, \ldots, X_n]$ generated by square free monomials. Then the natural map $\mathrm{GL}_r(R) \to \mathrm{GL}_r(R/J)$ is surjective.*

Given a simplicial subcomplex Σ of Δ_n and a ring A, let $I(\Sigma)$ be the ideal of $A[X_0, \ldots, X_n]$ generated by all square free monomials $X_{i_1} X_{i_2} \ldots X_{i_k}$ with $0 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $\{i_1, \ldots, i_k\}$ is not a face of Σ. By $A(\Sigma)$, we denote the discrete Hodge A-algebra $A[X_0, \ldots, X_n]/I(\Sigma)$.

The following result is due to Vorst ([6], Lemma 3.4) and is very crucial for the proof of our results.

Proposition 2.3 *Let Σ be a simplicial subcomplex of Δ_n which is not a simplex. Then there exists an $i \in \{0, 1, \ldots, n\}$ and simplicial subcomplexes $\Sigma_2 \subset \Sigma_1 \subset \Sigma$ such that we have a cartesian square of*

rings

$$\begin{array}{ccc} A(\Sigma) & \xrightarrow{i_1} & A(\Sigma_1) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ A(C(\Sigma_2)) & \xrightarrow[j_2]{} & A(\Sigma_2) \end{array}$$

where all maps are natural surjections and $\Sigma_2 \subset \Sigma_1 \subset \Sigma^i$, *where* Σ^i *is the* $n-1$ *simplex of which* i *is not a vertex and* $C(\Sigma_2)$ *is the cone on* Σ_2 *with vertex* i. *Note that* j_2 *is a split surjection and* $A(C(\Sigma_2)) = A(\Sigma_2)[X_i]$.

We end this section by stating two results of Wiemers ([7], Theorem 3.6) and ([8], Theorem 4.3) respectively which will be used in section 4.

Theorem 2.4 *Let* A *be a ring of dimension* d. *Let* $I \subset J$ *be ideals in* $R = A[X_1, \ldots, X_n]$ *generated by square free monomials. Let* P *be a projective module over* R/I. *If either rank* $P \geq d+1$ *or rank* $P \geq d$ *and* $1/d! \in A$, *then the natural map* $\text{Aut}_{R/I}(P) \to \text{Aut}_{R/J}(P/\overline{J}P)$ *with* $\overline{J} = J/I$ *is surjective.*

Theorem 2.5 *Let* A *be a ring of dimension* d *with* $1/d! \in A$ *and* $B = A[X_1, \ldots, X_n]$. *Let* P *and* P_1 *be projective* B*-modules of rank* $\geq d$. *Assume* $P \oplus B \xrightarrow{\sim} P_1 \oplus B$. *If* $P/(X_1, \ldots, X_n)P \xrightarrow{\sim} P_1/(X_1, \ldots, X_n)P_1$, *then* $P \xrightarrow{\sim} P_1$.

In other words, if the projective A*-module* $P/(X_1, \ldots, X_n)P$ *is cancellative, then* P *is cancellative.*

3 Main Theorem

In this section we prove our main results mentioned in the introduction.

Proof of Theorem 1.1 : Let $B = A[X_0, \ldots, X_n]/I$ be a discrete Hodge A-algebra and let P be a projective B-module of rank r (here I is a monomial ideal). It is enough to assume that I is a square free monomial ideal. Then $I = I(\Sigma)$ for some simplicial subcomplex Σ of Δ_n and $B = A(\Sigma)$. We will use induction on n.

If $n = 0$, then there is nothing to prove, as $A(\Sigma) = A$ or $A[X_0]$. Let $n > 0$ and assume the result for $n-1$. We will apply (2.3). By induction hypothesis, all projective modules of rank r over $A_1 = A(\Sigma_1)$ and $A_0 = A(\Sigma_2)$ are extended from A. Also all projective modules of rank r over $A_2 = A(C(\Sigma_2)) = A(\Sigma_2)[X_i]$ are extended from $A[X_i]$ and hence are extended from A.

Write $P_i = P \otimes_A A_i$, $i = 0, 1, 2$. Clearly, the natural map $\text{Aut}_{A_2}(P_2) \to \text{Aut}_{A_0}(P_0)$ is surjective. Hence, if $Q = P/(X_0, \ldots, X_n)P$, then $P_1 \xrightarrow{\sim} Q \otimes A_1$ and $P_2 \xrightarrow{\sim} Q \otimes A_2$, by induction hypothesis. Hence, by $(2.1(ii))$, $P \xrightarrow{\sim} Q \otimes A$, i.e. P is extended from A. This proves the result. $\square$

Proof of Theorem 1.2 : Let $R = A[X_0, \ldots, X_n]/I$ be a discrete Hodge A-algebra and let P be a projective R-module of rank r (here I is a monomial ideal). Again, it is enough to assume that I is a square free monomial ideal. Then $I = I(\Sigma)$ for some simplicial subcomplex Σ of Δ_n and $R = A(\Sigma)$. We will use induction on n.

When $n = 0$, there is nothing to prove as $R = A$ or $A[X_0]$. Let $n > 0$ and assume the result for $n-1$. We will apply (2.3). Let $A_1 = A(\Sigma_1)$, $A_2 = A(C(\Sigma_2))$ and $A_0 = A(\Sigma_2)$. Write $P_i = P \otimes_A A_i$ for $i = 0, 1, 2$.

Since $R \to A_i$ are natural surjections, $P_i \oplus A_i$ are extended from A, $i = 1, 2$. Hence, by induction hypothesis P_i is extended from A, $i = 1, 2$. Therefore, if $Q = P/(X_0, \ldots, X_n)P$, $P_i \xrightarrow{\sim} Q \otimes A_i$, $i = 1, 2$. Clearly, the natural map $\text{Aut}_{A_2}(P_2) \to \text{Aut}_{A_0}(P_0)$ is surjective. Hence, by (2.1(ii)), $P \xrightarrow{\sim} Q \otimes_A R$, i.e. P is extended from A. This proves the result. $\square$

Proof of Theorem 1.3 : It is enough to show that the natural map $\text{Um}_r(R) \to \text{Um}_r(R/J)$ is surjective for every ideal J of R generated by square free monomials.

Let $v \in \text{Um}_r(R/J)$. We have an exact sequence $0 \to P \to (R/J)^r \xrightarrow{v} R/J \to 0$.

Since $P \oplus A/J$ is free, by (1.2), P is extended from A, i.e. $P = \overline{P} \otimes_A R$, where $\overline{P} = P/(X_1, \ldots, X_n)P$. Hence, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{P} \otimes_A R \arrow[r] \arrow[d, "\tilde{\to}"] & (R/J)^r \arrow[r, "v(0)\otimes R"] \arrow[d, dashed, "\sigma"] & R/J \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & P \arrow[r] & (R/J)^r \arrow[r, "v"] & R/J \arrow[r] & 0
\end{tikzcd}$$

where $v(0)$ is the image of v in $\text{Um}_r(A)$ under the map $R/J \to A$ given by $\overline{X}_i \mapsto 0$, $i = 1, \ldots, n$. Hence, there exists $\sigma \in \text{GL}_r(R/J)$ such that $v\sigma = v(0) \otimes R$. By (2.2), σ can be lifted to $\Delta \in \text{GL}_r(R)$ and $v(0)\Delta^{-1} \in \text{Um}_r(R)$ is a lift of v. This proves the result. $\square$

4 Some Auxiliary Results

As an application of (2.3), we will give an alternative proof of the following result of Wiemers ([7], Corollary 4.4).

Theorem 4.1 *Let A be a ring of dimension d with $1/d! \in A$. Let $B = A[X_0, \ldots, X_n]/I$ be a discrete Hodge A-algebra. Let P be a projective B-module of rank $\geq d$. If the projective A-module $P/(X_0, \ldots, X_n)P$ is cancellative, then P is cancellative.*

Proof If B is a polynomial ring over A, then the result follows from (2.5). It is enough to assume that I is generated by square free monomials. Hence $I = I(\Sigma)$ for some simplicial subcomplex Σ of Δ_n. We will apply induction on n.

By (2.3), we have the following cartesian square

$$\begin{tikzcd}
A(\Sigma) \arrow[r, "i_1"] \arrow[d, "i_2"'] & A(\Sigma_1) \arrow[d, "j_1"] \\
A(C(\Sigma_2)) \arrow[r, "j_2"'] & A(\Sigma_2).
\end{tikzcd}$$

By (2.4), the natural map $\mathrm{Aut}_{A(C(\Sigma_2))}(P \otimes A(C(\Sigma_2))) \to \mathrm{Aut}_{A(\Sigma_2)}(P \otimes A(\Sigma_2))$ is surjective and by induction hypothesis on n, $P \otimes A(C(\Sigma_2))$ and $P \otimes A(\Sigma_1)$ are cancellative. Hence, by (2.1(ii)), P is cancellative. This proves the result. $\square$

Theorem 4.2 *Let A be a ring of dimension d with $1/d! \in A$ and $R = A[X_1, \ldots, X_n]$. Let P be a projective R-module of rank d such that $P \oplus R$ is extended from A. Then P is extended from A.*

Proof By Quillen's local-global principle ([3], Theorem 1), it is enough to assume that A is local. Then $P \oplus R$ is free. Since $P/(X_1, \ldots, X_n)P$ is free, by (2.5), P is free. This proves the result. $\square$

Remark 4.3 When P is stably free, the above result (4.2) is due to Ravi A Rao ([4] Corollary 2.5). More precisely, Rao proved that if A is a ring of dimension d with $1/d! \in A$, then every $v \in \mathrm{Um}_{d+1}(A[X])$ is extended from A, i.e. there exists $\sigma \in \mathrm{SL}_{d+1}(A[X])$ such that $v\sigma = v(0)$.

Following the proof of (1.2) and using (4.2), we get the following:

Corollary 4.4 *Let A be a ring of dimension d with $1/d! \in A$. Let B be a discrete Hodge A-algebra. Let P be a projective B-module of rank d such that $P \oplus B$ is extended from A, then P is extended from A. In particular, every stably free B-module of rank d is extended from A.*

During CAAG VII meeting in Bangalore, Kapil H Paranjape asked if we can extend the above results (1.1, 1.2, 4.4) for locally discrete Hodge A-algebras (Definition: A positively graded A-algebra B is a locally discrete Hodge A-algebra if $B_{\mathfrak{p}}$ is a discrete Hodge $A_{\mathfrak{p}}$-algebra for every prime ideal $\mathfrak{p}$ of A. The answer is yes and follows from the following result of Lindel ([1], Theorem 1.3) which generalises Quillen's patching theorem ([3], Theorem 1) from polynomial rings to positively graded rings.

Theorem 4.5 *Let A be a ring and let M be a finitely presented module over a positively graded ring $R = \oplus_{i \geq 0} R_i$, $R_0 = A$. Then the set $J(A, M)$, of all $u \in A$ for which M_u is extended from A_u, is an ideal of A.*

In particular, if $M_{\mathfrak{p}}$ is extended from $A_{\mathfrak{p}}$ for all prime ideal $\mathfrak{p}$ of A, then M is extended from A.

References

[1] Lindel H., *On projective modules over positively graded rings*, Vector bundles on algebraic varieties (Bombay, 1984), 251–273, Tata Inst. Fund. Res. Stud. Math., 11, Tata Inst. Fund. Res., Bombay, 1987.

[2] Milnor J., *Introduction to Algebraic K-Theory*, Annals of Math. Studies, Princeton Univ. Press, Princeton, 1971.

[3] Quillen D., *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167-171.

[4] Rao Ravi A., *The Bass-Quillen conjecture in dimension three but characteristic $\neq 2, 3$ via a question of A. Suslin*, Invent. Math. **93** (1988), 609-618.

[5] Roitman M., *On stably extended projective modules over polynomial rings*, Proc. AMS **97** (1986), 585-589.

[6] Vorst T., *The Serre problem for discrete Hodge algebras*, Math. Z. **184** (1983), 425-433.

[7] Wiemers A., *Some properties of projective modules over discrete Hodge algebras*, J. Algebra **150** (1992), 402-426.

[8] Wiemers A., *Cancellation properties of projective modules over Laurent polynomial rings*, J. Algebra **156** (1993), 108-124.